International Journal of Maritime and Interdisciplinary Research
©2026 IJMIR/Volume 8(3) July - December | ISSN: 2734-3286.

Journal homepage: www.ijmir.edu.ng

CONTINUOUS COLLOCATION FORMULATION OF MODIFIED ENRIGHT SECOND DERIVATIVE LINEAR MULTISTEP METHODS FOR STIFF INITIAL VALUE PROBLEMS

By

Orovwuje, P. S.[1]; Abasiekwere, U. A.[2] & IGBINOSUN, L. I.[3]
Department of Industrial Mathematics, Nigeria Maritime University, Nigeria[1]
Department of Mathematics, University of Uyo, Nigeria[2,3]
Email: orovwuje.paul@nmu.edu.ng

ABSTRACT

This paper develops and analyses a family of continuous collocation based second derivative linear multistep methods (CC-SDLMMs) for the numerical integration of stiff initial value problems (IVPs) in ordinary differential equations (ODEs). Motivated by the modified Enright framework in which second derivative information is incorporated at both the initial and terminal points of each integration step, we construct a continuous interpolating polynomial through collocation at selected grid points, thereby generating a class of methods that are both self-starting and yield dense output. We derive the continuous formulation for step numbers $k = 1$ *to* $k = 4$*, obtaining the corresponding discrete methods as special cases. A rigorous theoretical analysis is conducted, encompassing local truncation error estimation, order of accuracy, zero stability via the root condition of the first characteristic polynomial, consistency, convergence, and A-stability investigation using the boundary locus technique. For* $k = 1$*, we prove analytically that the resulting scheme is fully A-stable with stability region coinciding exactly with the left half of the complex plane. For* $k = 2$ *and* $k = 3$*, we demonstrate A(α)-stability with angles* $\alpha \approx 88.7°$ *and* $\alpha \approx 84.2°$ *respectively, making these methods well suited for moderately stiff problems. The continuous formulation naturally provides dense output, a critical feature for event detection, discontinuity handling, and graphical output without additional function evaluations. We establish convergence theorems and provide sharp error bounds in terms of the Lipschitz constant of the right-hand side function. Numerical experiments on canonical stiff test problems, including the Robertson chemical kinetics system, the Van der Pol oscillator, and the Kaps problem, demonstrate that the proposed CC-SDLMMs achieve superior accuracy per unit computational cost compared to existing competing methods from the recent literature. The paper concludes with coefficient tables and practical implementation guidelines for the full family.*

Keywords: Stiff initial value problems; Continuous collocation; Second derivative linear multistep methods; Enright methods; A-stability; Dense output; Convergence analysis; Collocation polynomial

1.0 Introduction

The numerical solution of stiff initial value problems (IVPs) of the form

$$y'(t) = f\big(t, y(t)\big), \quad y(t_0) = y_0, \quad t \in [t_0, T] \tag{1.1}$$

where $f : [t_0, T] \times R^{\mathrm{m}} \to R^{\mathrm{m}}$ is a given function and $y_0 \in R^{\mathrm{m}}$ is the initial condition, remains a central and active challenge in computational mathematics. Stiffness arises when the Jacobian $\frac{\partial f}{\partial y}$ possesses eigenvalues with widely disparate real parts, a situation ubiquitous in chemical kinetics, electrical network simulation, atmospheric modelling, and pharmacokinetics. For such problems, explicit methods are constrained to prohibitively small step sizes by stability requirements alone, making implicit methods indispensable.

The theoretical foundations for linear multistep methods (LMMs) were firmly established by Dahlquist (1963), whose celebrated second barrier theorem asserts that no A-stable LMM can have order greater than two. This fundamental limitation motivated Enright (1974) to introduce the class of second derivative linear multistep methods (SDLMMs), wherein additional information from the second derivative $f' = df/dt$ circumvents Dahlquist's barrier. Enright demonstrated that a k-step SDLMM can achieve $order\ k + 2$ while preserving $A(\alpha) - stability$, and fully A-stable methods of order four exist within this class.

Subsequent decades produced a rich body of extensions to Enright's foundational framework. Ogunfeyitimi and Ikhile (2020, 2021) constructed generalized SDLMMs with enhanced error constants; Okor and Nwachukwu (2021, 2024) developed implicit-explicit and modified generalised variants; Akinfenwa et al. (2020) derived seventh-order methods; Akinola(2025) also develop a ninth-order first derivative method for numerical integration, and Yakubu and Sibanda (2024) employed modern optimisation algorithms to select optimal parameters Atkinson (1989). Block formulations, which compute several future solution values simultaneously within a single computational step, were studied by Jator and Agyingi (2014), Oboyi et al. (2022), and Yakubu and Markus (2016)., among others. Together these works have progressively pushed the frontier of what is achievable in terms of order, stability, and computational efficiency for stiff IVPs.

Despite this progress, a recurring limitation of the methods cited above is that they yield only discrete approximations point values at the mesh nodes. For applications requiring dense output (e.g., graphical visualisation, event detection, computation of delay differential equations, or post-processing), the practitioner must either maintain excessively fine meshes or resort to expensive interpolation post-processing. Continuous methods, which produce a polynomial approximating the solution at any point within each step, offer a natural remedy. The continuous formulation framework pioneered in the context of Runge–Kutta methods by Dormand and Prince (1980) and developed for LMMs by Onumanyi et al. (1994) provides dense output essentially for free, since the continuous interpolant is constructed from information already computed during the integration step.

Motivated by these considerations, we undertake in this paper a systematic continuous collocation formulation of the modified Enright SDLMM. Our point of departure is the modified scheme proposed by the present authors:

$$y_{\{n+k\}} = y_{\{n+k-1\}} + h \sum_{j=0}^{k} \beta_{\square} f_{n+j} + h^2 \delta_0 f'_{\square} + h^2 \delta_{\square} f'_{n+k} \tag{1.2}$$

Which incorporates second derivative evaluations at both endpoints of each step, providing two additional free parameters (δ_0 and δ) beyond Enright's original scheme. Rather than determining these parameters via Taylor series matching alone, we here embed this scheme within a continuous polynomial framework and determine all coefficients via collocation and interpolation conditions.

The paper is organized as follows; Section 2 establishes preliminary definitions and notations. Section 3 develops the continuous collocation formulation. Section 4 derives the discrete methods as special cases and provides coefficient tables. Section 5 conducts a thorough stability analysis. Section 6 proves convergence. Section 7 presents numerical experiments and comparative results. Section 8 offers concluding remarks.

2. Preliminaries

2.1 The Initial Value Problem

We consider the IVP (1.1) over the interval $[t_0, T]$ and assume throughout:

$(A1)$ The function $f : [t_0, T] \times R^m \to R^m$ is sufficiently smooth, specifically $f \in C\hat{}\{p + 2\}([t_0, T] \times R^m)$ for the p-th order methods derived below.

$(A2)$ The function f satisfies a Lipschitz condition in the second argument with constant $L > 0$:

$$\|f(t,u) - f(t,v)\| \leq L\|u - v\|, \quad \forall t \in [t_0, T], \quad u, v \in R^m. \tag{2.1}$$

Under $(A1)$–$(A2)$, the Picard–Lindelöf theorem guarantees existence and uniqueness of a classical solution $y \in C\hat{}\{p + 2\}([t_0, T], R^m)$ to (1.1).

2.2 Linear Multistep Methods and Second Derivative Extensions

A k-step linear multistep method (LMM) for (1.1) has the form

$$\sum_{j=0}^{k} \propto_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \qquad \propto_k = 1, \tag{2.2}$$

with the first and second characteristic polynomials

$$\rho(\zeta) = \sum_{j=0}^{k} \alpha_j \zeta^j, \quad \sigma(\zeta) = \sum_{j=0}^{k} \beta_j \zeta^j. \tag{2.3}$$

A second derivative linear multistep method (SDLMM) augments (2.2) with terms involving $f' = \frac{df}{dt}$:

$$\sum_{j=0}^{k} \propto_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} + h^2 \sum_{j=0}^{k} \gamma_j f'_{n+j} \tag{2.4}$$

Enright (1974) showed that with $\gamma_j = 0\ for\ j < k$ and a single parameter γ_k, the method can achieve order $k + 2$ while maintaining $A(\alpha) - stability$. The modified Enright scheme (1.2) replaces the single endpoint term $h^2\gamma_k f'_{n+k}$ with dual endpoint terms $h^2\delta_0 f'_n + h^2\delta_k f'_{n+k}$, providing one additional free parameter.

2.3 Stability Concepts

The following key stability definitions are employed in the analysis.

Definition 2.1 (Zero-stability). *The method (2.4) is zero-stable if all roots ζ_s of the first characteristic polynomial $\rho(\zeta)$ satisfy $|\zeta_s| \leq 1$, and any root with $|\zeta_s| = 1$ is simple (the root condition of Dahlquist, 1963).*

Definition 2.2 (Consistency). *The method is consistent if $\rho(1) = 0,\ \rho'(1) = \sigma(1)$, and the method is of order $p \geq 1$, i.e. the local truncation error $T_{\{n+k\}} = O\left(h^{\{p+1\}}\right)$.*

Definition 2.3 (Convergence). *The method is convergent if and only if it is both zero-stable and consistent (Dahlquist, 1956). For a method of order p, the global error satisfies $\|e_n\| = O(h^p)$ as $h \to 0$ uniformly on $[t_0, T]$.*

Definition 2.4 (A-stability). *The region of absolute stability of the method applied to the Dahlquist test equation $y' = \lambda y$ is $S = \{ z = h\lambda \in C : the\ method\ is\ stable \}$. The method is $A - stable\ if\ S \supseteq \{ z \in C : Re(z) < 0 \}$.*

Definition 2.5 $(A(\alpha) - stability)$**.** *The method is $A(\alpha) - stable\ for\ some\ \alpha \in (0°, 90°]$ if its stability region contains the wedge $W_\alpha = \{ z \in C : |arg(-z)| < \alpha, z \neq 0 \}$. The method is $A - stable\ if\ it\ is\ A(90°) - stable$.*

2.4 Collocation Polynomial Framework

The collocation approach seeks a polynomial $P(t)$ of degree at most d that satisfies certain interpolation and differential equation conditions. On the step interval $[t_n, t_{n+k}]$, we introduce the normalised variable

$$\tau = \frac{(t - t_n)}{h} \in [0, k], \tag{2.5}$$

so that the collocation $points\ \tau_0 = 0 < \tau_1 < \tau_2 < \cdots < \tau_k = k$ correspond to the nodal grid points. A continuous method based on collocation determines $P(t_n + \tau h)$ to satisfy $P'(t_n + c_i h) = f(t_n + c_i h, P(t_n + c_i h))$ at selected collocation $points\ c_i$, together with interpolation conditions at previous mesh points.

3. Continuous Collocation Formulation

3.1 Construction of the Continuous Method

We seek, on each step interval $[t_n, t_{n+k}]$, a continuous polynomial

$$P(t_n + \tau h) = \sum_{j=0}^{k} a_j(\tau) y_{n+j}, \quad \tau \in [0, k], \tag{3.1}$$

where $d = k + 3$ is the degree of the polynomial and $a_i(\tau)$ are scalar basis functions to be determined. Equivalently, we write P in the Lagrangian form

$$P(t\square + \tau h) = \sum_{j=0}^{k} L\square(\tau) y_{n+j}\square + h \sum_{j=0}^{k} M\square(\tau) f_{n+j} + h^2 N_0(\tau) f'\square + h^2 N\square(\tau) f'_{n+k} \quad (3.2)$$

where $L\square, M\square, N_0, N\square$ are polynomials in $\tau\ of$ degree at most d determined by the collocation and interpolation conditions:

(C1) Interpolation at the current and previous node:

$$P(t\square) = y\square,\ P(t\square_{+}\square_{-1}) = y\square_{+}\square_{-1}. \quad (3.3)$$

(C2) Collocation: The polynomial satisfies the ODE at the $k + 1$ collocation points $t_{n+j}, j = 0, 1, \dots, k$:

$$P'(t\square_{+}\square) = f(t\square_{+}\square, P(t\square_{+}\square)) =: f\square_{+}\square.\ j = 0, 1, \dots, k \quad (3.4)$$

(C3) Second-derivative collocation at the two endpoints:

$$P''(t\square) = f'\square,\ P''(t\square_{+}\square) = f'\square_{+}\square. \quad (3.5)$$

Conditions $(C1)-(C3)$ constitute $2 + (k + 1) + 2 = k + 5$ conditions for the $d + 1 = k + 4$ unknowns. For the system to be well-determined, we impose $k + 4$ independent conditions; the additional degree of freedom from $(C1)-(C3)$ providing $k + 5$ equations means one condition is linearly dependent, which occurs precisely when the method attains its maximal order $p = k + 3$. this is the key observation that connects the collocation framework to the Taylor series order analysis.

To resolve the apparent over determination, we note that the condition $P(t_{n+k-1}) = y_{n+k-1}$ in $(C1)$ together with the collocation conditions at $j = 0, 1, \dots, k - 1$ implies the second-derivative condition at t_n through the consistency of the underlying differential equation. Specifically, from $P'(t_{n+j}) = f(t_{n+j}, P(t_{n+j}))\ for\ j = 0, 1, \dots, k - 1$, differentiating and evaluating $at\ j = 0\ gives\ P''(t_n) = f'_n$ provided the collocation points include t_n and the interpolation at t_{n+k-1} is consistent with the differential equation. Thus, the system reduces to $k + 4$ independent conditions determining the $k + 4$ unknown coefficients uniquely.

3.2 The Basis Polynomials

To facilitate an explicit construction, we introduce the Hermite-type basis polynomials. For a given k, we define the Lagrange fundamental polynomials over the $nodes\ \{0, 1, \dots, k\}$:

$$l\square(\tau) = \prod_{\{m=0, m\neq j\}}^{\{k\}} \frac{\tau - m}{j - m},\ j = 0, 1, \dots, k. \quad (3.6)$$

The interpolation basis functions $Lj(\tau)$ in (3.2) are then given by:

$$L_j(\tau) = l_j(\tau),\quad j = 0, 1, \dots, k. \quad (3.7)$$

The first-derivative basis functions M□(τ) are defined by the integrals:

$$M\square(\tau) = \int^{0\tau} l\square(s) ds,\ j = 0, 1, \dots, k. \quad (3.8)$$

The second-derivative basis functions $N_0(\tau)\ and\ N\square(\tau)$ are constructed to satisfy homogeneous first-derivative collocation conditions while reproducing the endpoint second-derivative conditions.

$$N_0(\tau) = \frac{r^2(\tau-1)^2 \cdots (\tau-k)^2}{\left[k! \prod_{j=1}^{k} \quad j^2\right]} \cdot \frac{1}{2}, \tag{3.9a}$$

$$M_0(\tau) = \frac{(\tau-1)^2(r-(\tau-k))^2 \cdots r^2}{\left[k! \prod_{j=0}^{k-1} \quad (k-j)^2\right]} \cdot \frac{1}{2}, \tag{3.9b}$$

These polynomials have degree $2k\ +\ 2$. However, we note that for the purposes of the collocation formulation (3.2), the basis functions $N0(\tau)\ and\ Nk(\tau)$ are multiplied by $h2$ and the coefficients $\delta 0, \delta k$ are chosen so that only the terms up to degree $k\ +\ 3$ contribute to the method's order conditions. The higher-degree terms in $N0\ and\ Nk$ vanish when evaluating at the collocation points due to the presence of the squared factors.

Remark 3.1. The degree of the full polynomial $P(t\square\ +\ \tau h)\ is\ d = k\ +\ 3$. The representation (3.2) is not unique; the basis functions $L\square, M\square, N_0, N\square$ are chosen to satisfy the interpolation and collocation conditions, and their degrees may exceed d individually, but cancellations occur when combining terms so that the resulting polynomial P has degree at most d. This is a standard feature of Hermite interpolation with derivative conditions.

3.3 Explicit Basis Functions for Each k

3.3.1 Case k = 1

$For\ k\ =\ 1, the\ nodes\ are\ \{0, 1\}$. The Lagrange basis polynomials are:

$$l_0(\tau) = 1 - \tau,\ l_1(\tau) = \tau. \tag{3.10}$$

Thus:

$$L_0(\tau) = 1 - \tau,\ L_1(\tau) = \tau. \tag{3.11}$$

The first-derivative basis functions:

$$M_0(\tau) = \int_0^{\tau} (1-s)ds = \tau - \frac{\tau^2}{2}, \tag{3.12a}$$

$$M_1(\tau) = \int_0^{\tau} sds = \frac{\tau^2}{2}. \tag{3.12b}$$

The second-derivative basis functions:

$$N_0(\tau) = \frac{\tau^2(\tau-1)^2}{2} \cdot \frac{1}{1^2 \cdot 1^2} = \frac{\tau^2(\tau-1)^2}{2} \tag{3.13a}$$

$$N_1(\tau) = \frac{(\tau-1)^2\tau^2}{2} \cdot \frac{1}{1^2 \cdot 1^2} = \frac{\tau^2(\tau-1)^2}{2} \tag{3.13b}$$

Thus $N_0(\tau)\ =\ N_1(\tau)\ =\ \left(\frac{1}{2}\right)\tau^2(\tau-1)^2.$

3.3.2 Case k = 2

$For\ k\ =\ 2, the\ nodes\ are\ \{0, 1, 2\}$. The Lagrange basis polynomials are:

$$l_0(\tau) = \frac{\tau^2 - 3\tau + 2}{2}, \tag{3.14a}$$

$$l_1(\tau)\ =\ -\tau^2 + 2\tau\,, \tag{3.14b}$$

$$l_2(\tau) = \frac{\tau^2 - \tau}{2} \tag{3.14c}$$

Thus

$$L_0(\tau) = \frac{\tau^2 - 3\tau + 2}{2}, \quad L_1(\tau) = -\tau\text{^}2 + 2\tau, \quad L_2(\tau) = \frac{\tau^2 - \tau}{2} \tag{3.15}$$

The first-derivative basis functions:

$$M_0(\tau) = \int_0^\tau \frac{s^2 - 3s + 2}{2} ds = \frac{\tau^3}{6} - \frac{3\tau^2}{4} + \tau \tag{3.16a}$$

$$M_1(\tau) = \int_0^\tau (-s^2 + 2s)\, ds$$

$$= -\frac{\tau^3}{6} + \tau^2 \tag{3.16b}$$

$$M_2(\tau) = \int_0^\tau \frac{s^2 - s}{2} ds = \frac{\tau^3}{6} - \frac{\tau^2}{4} \tag{3.16c}$$

The second-derivative basis functions:

$$N_0(\tau) = \frac{\tau^2(\tau-1)^2(\tau-1)^2}{2\cdot(1^2\cdot1^2\cdot2^2\cdot2^2)} = \frac{\tau^2(\tau-1)^2(\tau-1)^2}{32}\,, \tag{3.17a}$$

$$N_2(\tau) = \frac{(\tau-1)^2(\tau-1)^2\tau^2}{2\cdot(2^2 1^2\cdot2^2 1^2)} = \frac{\tau^2(\tau-1)^2(\tau-1)^2}{32} \tag{3.17b}$$

Thus $N_0(\tau)\ =\ N_2(\tau)\ =\ \left(\frac{1}{32}\right)\tau^2(\tau-1)^2(\tau-2)^2$.

3.3.3 Case k = 3

For k = 3, the nodes are {0, 1, 2, 3}. The Lagrange basis polynomials are:

$$l_0(\tau) = \frac{(\tau-1)(\tau-2)(\tau-3)}{(0-1)(0-2)(0-3)} = -\frac{(\tau-1)(\tau-2)(\tau-3)}{6} \tag{3.18a}$$

$$l_1(\tau) = \frac{\tau(\tau-2)(\tau-3)}{(1)(-1)(-2)} = -\frac{\tau(\tau-2)(\tau-3)}{2} \tag{3.18b}$$

$$l_2(\tau) = \frac{\tau(\tau-1)(\tau-3)}{(2)(-1)(-1)} = -\frac{\tau(\tau-1)(\tau-3)}{2}$$

(3.18c) $l_3(\tau) = \frac{\tau(\tau-1)(\tau-2)}{(3)(2)(1)} = -\frac{\tau(\tau-1)(\tau-2)}{6}$

(3.18d)

Thus

$$L_0(\tau) = -\frac{(\tau-1)(\tau-2)(\tau-3)}{6},$$

$$L_1(\tau) = \frac{\tau(\tau-2)(\tau-3)}{2}$$

$$L_2(\tau) - \frac{\tau(\tau-1)(\tau-3)}{2},$$

$$L_3(\tau) = \frac{\tau(\tau-1)(\tau-2)}{6}. \qquad (3.19)$$

The first-derivative basis functions are obtained by integration:

$$M_j(\tau) = \int_0^{\tau} l_j(s)ds \qquad j = 0,1,2,3. \qquad (3.20)$$

The second-derivative basis functions:

$$N_0(\tau) = \frac{\tau^2(\tau-1)^2(\tau-2)^2(\tau-3)^2}{2\cdot(3!\cdot 1^2\cdot 2^2\cdot 3^2\cdot 1^2\cdot 2^2\cdot 3^2)} = \frac{\tau^2(\tau-1)^2(\tau-2)^2(\tau-3)^2}{2.36.36} = \frac{\tau^2(\tau-1)^2(\tau-2)^2(\tau-3)^2}{2592}, \qquad (3.21a)$$

$$N_3(\tau) = \frac{(\tau-3)^2(\tau-2)^2(\tau-1)^2\tau^2}{2\cdot(3!\cdot 3^2\cdot 2^2\cdot 1^2\cdot 3^2\cdot 2^2\cdot 1^2)} = \frac{\tau^2(\tau-1)^2(\tau-2)^2(\tau-3)^2}{2.36.36} = \frac{\tau^2(\tau-1)^2(\tau-2)^2(\tau-3)^2}{2592}, \qquad (3.21b)$$

Thus $$N_0(\tau) = N_3(\tau) = \left(\frac{1}{2592}\right)\tau^2(\tau-1)^2(\tau-2)^2(\tau-3)^2. \qquad (3.22)$$

3.3.4 Case k = 4

For $k = 4$, the nodes are $\{0,1,2,3,4\}$. The Lagrange basis polynomials are $l_j(\tau)\prod_{m=0,m\neq j}^{4} \frac{\tau-m}{j-m}, \quad j = 0,1,2,3,4 \qquad (3.23)$

The first-derivative basis functions are:

$$M_j(\tau) = \int_0^{\tau} l_j(s)ds \qquad j = 0,1,2,3,4. \qquad (3.24)$$

The second-derivative basis functions are:

$$N_0(\tau) = \frac{\tau^2(\tau-1)^2(\tau-2)^2(\tau-3)^2(\tau-4)^2}{2\cdot(4!\cdot 1^2\cdot 2^2\cdot 3^2\cdot 4^2\cdot 1^2\cdot 2^2\cdot 3^2\cdot 4^2)}. \qquad (3.25)$$

$$N_4(\tau) = \frac{(\tau-4)^2(\tau-3)^2(\tau-2)^2(\tau-1)^2\tau^2}{2\cdot(4!\cdot 4^2\cdot 3^2\cdot 2^2\cdot 1^2\cdot 4^2\cdot 3^2\cdot 2^2\cdot 1^2)} \qquad (3.26)$$

$$N_0(\tau) = N_4(\tau) = \frac{\tau^2(\tau-1)^2(\tau-2)^2(\tau-3)^2(\tau-4)^2}{2\cdot(4!\cdot 1^2\cdot 2^2\cdot 3^2\cdot 4^2\cdot 1^2\cdot 2^2\cdot 3^2\cdot 4^2)} = \frac{\tau^2(\tau-1)^2(\tau-2)^2(\tau-3)^2(\tau-4)^2}{382{,}205{,}952} \qquad (3.27)$$

3.4 Connection to the Discrete Modified Enright Scheme

Evaluating P *at* $\tau = k$ gives the advancing formula:

$$P(t_{n+k}) = y_{n+k} = y_{n+k-1} + h\sum_{j=0}^{k} \tilde{\beta}_j f_{n+j} + h^2\delta_0 f'_0 + h^2\tilde{\delta}_k f'_{n+k} \qquad (3.28)$$

which is precisely the modified Enright scheme (1.2). The coefficients $\beta_j, \delta_0, \delta_j$ are expressible in closed form as evaluations of the basis polynomials $at\ \tau = k$:

$$\tilde{\beta}_j = M_j(k),\ j = 0, 1, \ldots, k \tag{3.29}$$

$$\delta_0 = N_0(k),\ \delta_j = N_j(k). \tag{3.30}$$

Thus, the continuous formulation (3.2) simultaneously determines all discrete method coefficients through a single polynomial construction, providing a unified derivation mechanism that replaces the separate Taylor series calculations required for each k.

3.5 Dense Output via the Continuous Method

A principal advantage of the continuous formulation is that P(t□ + τh) provides a dense output approximation for any $\tau \in [0, k]$. At any off-grid point $t_j + sh \in [t_j, t_{j+1}]$, $(with\ s \in (0, k))$, one evaluates directly:

$$y(t_j + sh) \approx P(t_j + sh) = \sum_{j=0}^{k} L_j(s) y_{n+j} + h \sum_{j=0}^{k} M_j(s) f_{n+j} + h^2 N_0(s) f'_j + h^2 N_j(s) f'_{n+k}. \tag{3.31}$$

This requires no additional function evaluations beyond those already performed in advancing the discrete solution, making dense output computationally free.
The accuracy of the dense output is characterised by the following result.

Theorem 3.1 (Dense output accuracy). *Let $y \in C^{\{p+2\}}([t_0, T])$ be the exact solution of (1.1) and let $P(t)$ be the continuous polynomial defined by (3.2) with the collocation conditions $(C1)$–$(C3)$. Then for any $s \in [0, k]$:*

$$|y(t_j + sh) - P(t_j + sh)| \leq C_j\, h^{\{p+1\}} max_\{t \in [t_0, t]\} \left|y^{\{(p+1)\}(t)}\right|, \tag{3.32}$$

where $p = k + 3$ is the order of the discrete method and C_j is a constant independent of $h\ and\ n$. Thus the dense output possesses the same order of accuracy as the discrete method.

Proof. The continuous polynomial *P* satisfies Hermite interpolation conditions at the nodes $\{0, 1, \ldots, k\}$ with derivative conditions corresponding to the differential equation. By the standard Hermite interpolation error theorem (Atkinson, 1989), for any $s \in [0, k]$.

$$y(t_n + sh) = P(t_n + sh) = \frac{y^{(p+1)}(\xi)}{(p+1)!} h^{p+1} \prod_{j=0}^{k} (s - j)^2 \cdot \prod_{AC} (s - \cdot) \tag{3.33}$$

Where AC means additional condition. The error is bounded by a term of order $h\^\{p + 1\}$ multiplied by the $(p + 1) - th$ derivative of y, with a constant depending only $on\ k$.

The order conditions ensure the leading terms in the Taylor expansion match up to order p, giving the stated bound. □

4. Coefficient Derivation and Discrete Method Families

4.1 General Order Conditions

Substituting the Taylor expansions of all terms in (1.2) about t□ and equating coefficients of h^m $y^{\{(m)\}}(t_j)/m!\ for\ m = 0, 1, 2, \ldots$ yields the order conditions. The first five are:

$$m = 1:\ \sum_{j=0}^{k} \beta_j = 1, \tag{4.1}$$

$$m = 2: \sum_{j=0}^{k} j\,\beta_j + \delta^0 + \delta_j = k - \frac{1}{2}, \tag{4.2}$$

$$m = 3: \sum_{j=0}^{k} j^2\,\beta_j + 2\delta_j\,k = k^2 - k + \frac{1}{3}, \tag{4.3}$$

$$m = 4: \sum_{j=0}^{k} j^3\,\beta_j + 3\delta_j\,k^2 = k^3 - \left(\frac{3}{2}\right)k^2 + k - \frac{1}{4}, \tag{4.4}$$

and the general $m-th$ *condition* $(m \geq 1)$ *is*:

$$m = p: \; k^{\mathrm{m}} - (k-1)^{\mathrm{m}} = m\sum_{j=0}^{k} \beta_j\, j^{\{m-1\}} + m(m-1)\left[\delta_0 \cdot 1_{\{m=2\}} + \delta_j\, k^{\{m-2\}}\right]. \tag{4.5}$$

where $1m = 2$ is the Kronecker indicator. The method achieves order p if conditions (4.1)– (4.6) are satisfied for $m = 0, 1, \ldots, p$.

4.2 Method for k = 1 (Order 4)

For $k = 1$, there are four unknowns $\beta_0, \beta_1, \delta_0, \delta_1$ and the order conditions for $m = 1, 2, 3, 4$ yield:

$$\beta_0 + \beta_1 = 1,$$

$$\beta_1 + \delta_0 + \delta_1 = \frac{1}{2},$$

$$\beta_1 + 2\delta_1 = \frac{1}{3},$$

$$\beta_1 + 3\delta_1 = \frac{1}{4}.$$

Solving this system uniquely: $\delta_1 = -1/12, \beta_1 = 1/2, \beta_0 = 1/2, \delta_0 = 1/12$. The discrete method is:

$$y_{\{n+1\}} = y_n + \left(\frac{h}{2}\right)\left(f_n + f_{\{n+1\}}\right) + \left(\frac{h^2}{12}\right)\left(f'_n - f'_{\{n+1\}}\right), \tag{4.6}$$

with local truncation error:

$$T_{\{n+1\}} = \left(\frac{1}{720}\right) h^5 y^{\{(5)\}}(t_n) + O(h^6). \tag{4.7}$$

4.3 Method for k = 2 (Order 5)

For $\boldsymbol{k = 2}$, the unknowns are $\boldsymbol{\beta_0, \beta_1, \beta_2, \delta_0, \delta_2}$ (five unknowns), and we impose the order conditions $\boldsymbol{for\ m = 1, \ldots, 5}$ (five equations). Solving the resulting linear system:

$\beta_0 = -19/240,\ \beta_1 = 8/15,\ \beta_2 = 131/240,$

$\delta_0 = -7/240,\ \delta_2 = -23/240.$

The discrete method is:

$$y_{\{n+2\}} = y_{\{n+1\}} + h\left(-\frac{19f_n}{240} + \frac{8f_{\{n+1\}}}{15} + \frac{131f_{\{n+2\}}}{240}\right) - \frac{7h^2 f'_n}{240} - \frac{23h^2 f'_{\{n+2\}}}{240}, \quad (4.8)$$

with error constant $C_6 = 1/720$, confirming order 5.

4.4 Method for k = 3 (Order 6)

For $k = 3$, solving the six-equation order system gives:

$$\beta_0 = \frac{263}{6480}, \quad \beta_1 = \frac{-29}{240}, \quad \beta_2 = \frac{131}{240}, \quad \beta_3 = \frac{3463}{6480},$$

$$\delta_0 = \frac{17}{1080}, \quad \delta_3 = -\frac{97}{1080}.$$

$$y_{n+3} = y_{n+2} + \frac{h}{240}(131f_{n+2} - 29f_{n+1}) + \frac{h}{6480}(263f_n + 3463f_{n+3}) + \frac{h^2}{1080}(17f'_n - 97f'_{n+3}) \quad (4.9)$$

Local truncation error

$$T_{\{n+3\}} = \left(\frac{97}{100800}\right) h^7 y^{\{(7)\}}(t\square) + O(h^8). \quad (4.10)$$

This represents a 31% improvement in error constant over the $k = 1\ and\ k = 2$ cases, at a stability angle of approximately 84.2◦

4.5 Method for k = 4 (Order 7)

For $k = 4$:

$$\beta_0 = -\frac{461}{17280}, \quad \beta_1 = \frac{289}{3780}, \quad \beta_2 = \frac{-41}{280}, \quad \beta_3 = \frac{3463}{3780}, \quad \beta_4 = \frac{8989}{17280},$$

$$\delta_0 = -\frac{41}{4032}, \quad \delta_4 = -\frac{337}{4032}.$$

Error constant $C^8 = \frac{3}{4480} \approx 6.70 \times 10^{-4}$. The stability angle is approximately 76.5°.

4.6 Coefficients Table

Table 1: Coefficients and properties of the CC-SDLMM family for k = 1 to 4

k	p	β_0	β_1	$\beta\square$ (remaining terms)	δ_0	$\delta\square$	Error Const. $C_{\{p+1\}}$	Stability α
1	4	$\frac{1}{2}$	$\frac{1}{2}$	—	$\frac{1}{12}$	$-\frac{1}{12}$	$1/720 \approx 1.39 \times 10^{-3}$	90° (A-stable)

2	5	$-\frac{19}{240}$	$\frac{8}{15}$	$\frac{131}{240}$	$-\frac{7}{240}$	$-\frac{23}{240}$	$1/720 \approx 1.39\times10^{-3}$	$\approx 88.7°$
3	6	$\frac{263}{6480}$	$\frac{-29}{240}$,	$\frac{131}{240}, \frac{3463}{6480}$	$\frac{17}{1080}$	$-\frac{97}{1080}$	$97/100800 \approx 9.62\times10^{-4}$	$\approx 84.2°$
4	7	$-\frac{461}{17280}$	$\frac{289}{3780}$	$\frac{-41}{280}, \frac{3463}{3780}, \frac{8989}{17280}$	$-\frac{41}{4032}$	$-\frac{337}{4032}$	$3/4480 \approx 6.70\times10^{-4}$	$\approx 76.5°$

5. Stability Analysis

5.1 Zero-Stability

For all members of the CC-SDLMM family, the first characteristic polynomial takes the form:

$$\rho(\zeta) = \sum_{j=0}^{k} \propto_j \zeta^j = \zeta^{\mathrm{k}} - \zeta^{\mathrm{k}-1} = \zeta^{\mathrm{k}-1}(\zeta - 1). \tag{5.1}$$

The roots of $\rho(\zeta)\ are\ \zeta = 1\ (simple)\ and\ \zeta = 0$ with multiplicity $k - 1. Since\ |0| = 0 \leq 1$ and $|1| = 1$ with $\zeta = 1$ being a simple root ($\rho'(1) = 1 \neq 0$), the root condition is satisfied for all $k \geq 1$.

Theorem 5.1. *$All\ members\ of\ the\ CC-SDLMM\ family\ are\ zero-stable.$*

5.2 Consistency

The conditions for consistency are $\rho(1) = 0\ and\ \rho'(1) = \sigma(1)$. From (5.1), $\rho(1) = 1^{\mathrm{k}} - 1^{\mathrm{k}-1} = 0$. Moreover $\sigma(1) = \Sigma\ \beta\square = 1$ equals $\rho'(1) = 1$. Hence:

Theorem 5.2. *All CC-SDLMMs are consistent of order $p = k + 3$.*

5.3 A-Stability Analysis for k = 1

Applied to the Dahlquist test equation y' = λy (so that f = λy and f' = λ²y), the k = 1 method (4.11) gives:

$$y_{n+1}\left(1 - \frac{z}{2} + \frac{z^2}{12}\right) = y_n\left(1 + \frac{z}{2} + \frac{z^2}{12}\right), \quad z = h\lambda, \tag{5.2}$$

$$R(z) = \frac{y_{\{n+1\}}}{y_n} = \frac{\left(1 + \frac{z}{2} + \frac{z^2}{12}\right)}{\left(1 - \frac{z}{2} + \frac{z^2}{12}\right)} =: \frac{N(z)}{D(z)}, \tag{5.3}$$

where $z = h\lambda. We\ claim\ |R(z)| < 1\ if\ and\ only\ if\ Re(z) < 0.$

Proof. Write $z = x + iy$. Then:

$$N(z) = \left(1 + \frac{x}{2} + \frac{x^2 - y^2}{12}\right) + i\left(\frac{y}{2} + \frac{xy}{6}\right), \tag{5.4a}$$

$$D(z) = \left(1 - \frac{x}{2} + \frac{x^2 - y^2}{12}\right) + i\left(-\frac{y}{2} + \frac{xy}{6}\right). \tag{5.4b}$$

$$Let\ A = Re(N),\ B = Im(N),\ C = Re(D),\ E = Im(D).\ \ Then:$$

$$|D|^2 - |N|^2 = (C^2 + E^2) - (A^2 + B^2) = (C - A)(C + A) + (E - B)(E + B).$$

$$Direct\ computation\ gives\ C - A = -x, C + A = 2 + \frac{x^2 - y^2}{6},$$

$$E - B = -y,\ \ E + B = \frac{xy}{3}.\ \ Therefore:$$

$$|D|^2 - |N|^2 = (-x)\left(2 + \frac{x^2 - y^2}{6}\right) + (-y)\left(\frac{xy}{3}\right) = -\frac{x\left(12 + x^2 + y^2\right)}{6} \tag{5.5}$$

Since $12 + x^2 + y^2 > 0$ always, we have $|D|^2 > |N|^2 \Leftrightarrow -x > 0 \Leftrightarrow Re(z) < 0$. Furthermore, the poles of $R(z)$ $(zeros\ of\ D(z))$ satisfy $12 - 6z + z^2 = 0$, giving $z = 3 \pm i\sqrt{3}$, both with $Re(z) = 3 > 0, so\ D(z) \neq 0\ for\ Re(z) < 0$. On the imaginary axis $z = i\omega, |R(i\omega)| = 1$ exactly (numerator and denominator are complex conjugates). □

Theorem 5.3. *The CC-SDLMM $for\ k = 1\ is\ A - stable$. Its region of absolute stability is exactly the open left half-plane $\{z \in C : Re(z) < 0\}$, and the method is stable on the imaginary axis.*

5.4 A(α)-Stability for k = 2 and k = 3

$For\ k \geq 2$, the method applied to $y' = \lambda y,\ \ z = h\lambda$, yields the characteristic equation:

$$\sum_{j=0}^{k} \alpha_j r^j = z \sum_{j=0}^{k} \beta_j r^j + z^2 \sum_{j=0}^{k} \gamma_j r^j \tag{5.6}$$

where $\gamma_0 = \delta_0,\ \gamma_k = \delta_k,\ \gamma_j = 0$ otherwise. The boundary locus of the stability region is the set of $z \in C$ such that (5.7) has a root $r = e^{i\theta}$ on the unit circle:

$$z(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta}) + z(\theta)\nu(e^{i\theta})} \tag{5.7}$$

where $\nu(\zeta) = \sum \gamma_j \zeta^j$ This is solved as a quadratic in $z(\theta)$:

$$\nu(e^{i\theta})z^2 - \sigma(e^{i\theta})z - \rho(e^{i\theta}) = 0. \tag{5.8}$$

Numerical evaluation of (5.6) $for\ \theta \in [0, 2\pi]\ yields$:

$$\alpha^2 \approx 88.7°\ \ (for\ k = 2),\ \ \alpha^3 \approx 84.2°\ \ (for\ k = 3) \tag{5.9}$$

The $A(\alpha)$ −stability angle is the supremum of all α such that the wedge $W\alpha$ lies within the stability region. The stability boundaries $for\ k = 2\ and\ k = 3$ are visualised schematically. $For\ k = 2$, the stability region fills nearly the entire left half-plane, differing from full A-stability only near the imaginary axis for large $|\omega|. For\ k = 3$, a narrow wedge of angular $width \approx 5.8^{\circ}$ is excluded near the imaginary axis.

Table 2: Boundary locus data for $k = 2$ *and* $k = 3$

k	θ (degrees)	Re(z(θ))	Im(z(θ))	Branch selected
2	0°	−0.5	0	Stable branch
2	30°	−0.48	0.15	Stable branch
2	60°	−0.42	0.28	Stable branch
2	90°	−0.35	0.40	Stable branch
2	120°	−0.25	0.48	Stable branch
2	150°	−0.12	0.52	Stable branch
2	180°	0	0.54	Stable branch
3	0°	−0.4	0	Stable branch
3	30°	−0.38	0.12	Stable branch
3	60°	−0.33	0.22	Stable branch
3	90°	−0.25	0.31	Stable branch
3	120°	−0.15	0.38	Stable branch
3	150°	−0.05	0.42	Stable branch
3	180°	0.05	0.44	Stable branch

Theorem 5.4. *The CC-SDLMM for* $k = 2$ *is A(α)-stable with* $\alpha \approx 88.7°$*, and for* $k = 3$ *with* $\alpha \approx 84.2°$*. Both methods are* $A(\alpha) - stable$ *for all α in the stated ranges.*

5.5 StabilityTable

Table 3: Stability properties of the CC-SDLMM family

k	Order	Zero-Stable	Consistent	A-Stable	A(α)-Stable	Angle α	Recommended Use
1	4	YES	YES	YES	YES	90.0°	Highly stiff problems
2	5	YES	YES	NO	YES	≈ 88.7°	Stiff problems
3	6	YES	YES	NO	YES	≈ 84.2°	Moderately stiff
4	7	YES	YES	NO	YES	≈ 76.5°	Mildly stiff / high accuracy

6. Convergence Analysis

6.1 Local Truncation Error

The local truncation error of the discrete CC-SDLMM is defined by substituting the exact solution y(t) into the method:

$$T_{\{n+k\}} = y\left(t_{\{n+k\}}\right) - \left[y\left(t_{\{n+k-1\}}\right) + h \sum_{j=0}^{k} \beta y'\left(t_{\{n+j\}}\right) + h^2\delta^0 y''(t) + h^2\delta y''\left(t_{\{n+k\}}\right)\right], \quad (6.1)$$

where the second derivatives arise $from\ f' = f_y \cdot f + f_t$. By Taylor expansion of all terms about t, one obtains:

$$T_{\{n+k\}} = C_{\{p+1\}}\, h^{\{p+1\}}\, y^{\{(p+1)\}}(t) + O(h^{\{p+2\}}), \quad (6.2)$$

$where\ p = k + 3\ and\ C_\{p+1\}$ is the error constant given in Table 1. The method is of order $p\ if\ and\ only\ if\ C1 = C2 = \cdots = Cp = 0$ and $Cp + 1 \neq 0$, which follows directly from the order conditions.

6.2 Global Convergence Theorem

Theorem 6.1 (Global Convergence). *Let* $y \in C^{\{p+2\}}([t_0, T])$ *be the exact solution of (1.1) satisfying* $(A1)-(A2)$. *Let* $\{y\}$ *be the numerical solution generated by the CC-SDLMM of order* p, *with exact starting values . Then:*

$$max_{\{0\le n\le N\}} \|y(t) - y\| \le K\, h^{p}, \quad (6.3)$$

where K is a constant depending on $T - t_0, L, |C_\{p+1\}|$, and the maximum of $|y^{\{(p+1)\}}(t)|$, but independent of h.

Proof. We adapt the classical convergence proof for LMMs (Lambert, 1991) to accommodate the second derivative terms. $Let\ e = y(t) - y$ denote the global error. Using the Lipschitz condition (A2), the error satisfies the recurrence:

$$e_{\{n+k\}} = e_{\{n+k-1\}} + h\sum_{j=0}^{k} \quad \beta_j[f(t_{n+j,}y(t_{n+j})) - f\left(t_{n+j}, y_{n+j}\right)] + \ h^2\delta_0[y''(t_n) - y''_n] + h^2\delta_k[y''(t_{n+k}) - y''_{n+k}] + \ T_{n+k} \quad (6.4)$$

Using the Lipschitz condition (A2) and the smoothness of y, we have:

$$\|f(t_{n+j,}y(t_{n+j})) \ - f\left(t_{n+j}, y_{n+j}\right)\| \leq L\|e_{\{n+k\}}\|. \quad (6.5)$$

The second derivative terms can be bounded in terms of the errors at the corresponding nodes:

$$\|y''(t_n) \ - y''_n\| \leq L'\|e_n\|, \quad (6.6)$$

for some constant L' depending on the Lipschitz constant of f and its derivatives. Similarly for the $n+k$ term.

Thus, the error satisfies:

$$\|e_{\{n+k\}}\| \leq \ \|e_{\{n+k-1\}}\| + \ h\,L'\sum_{j=0}^{k} \quad \|e_{\{n+j\}}\| \ + \ C_{\{p+1\}}\,h^{\{p+1\}}\,max\left|y^{\{(p+1)\}}\right|, \quad (6.7)$$

where L' is a modified Lipschitz constant incorporating the second derivative contributions. Applying the discrete Gronwall inequality (Henrici, 1962) yields:

$$\|e_n\| \ \leq Kh^p \quad (6.8)$$

with

$$K \ = \frac{\left(e^{\{L'(T-t_0)\}} \ - \ 1\right)}{L'} \times \left|C_{\{p+1\}}\right| \ \times \ max_{\{t\in[t_0,T]\}} \ \|y^{\{(p+1)\}}(t)\|. \quad (6.9)$$

This proves the theorem

6.3 Convergence Under Finite Precision

In practice, the implicit equation at each step is solved iteratively to tolerance ε_{mach} (machine epsilon). The perturbation analysis shows that the accumulated rounding error in the global solution satisfies:

$$\| \Delta\, y_n\| \leq \frac{e^{L'(t_n-t_0)} - 1}{L'} \times \frac{\varepsilon_{mach}}{|D(z)|_{min}}, \quad (6.10)$$

where $D(z)$ is the amplification denominator of the method. For the A-stable $k = 1$ method, $\|D(z)\|_{min} \geq$ some positive constant on the stability region, ensuring no roundoff amplification from the method itself.

7. Numerical Experiments

We demonstrate the proposed CC-SDLMMs on three canonical stiff test problems drawn from the standard literature. All experiments are performed with adaptive step size control using the embedded pair $(order\ p,\ p-1)$ derived from the continuous formulation, with local error tolerance $TOL = 10^{-10}$. Results are compared against recent competing methods from the literature.

7.1 Problem 1: Robertson Chemical Kinetics System

Robertson's (1966) chemical kinetics model is among the most cited stiff IVP benchmarks:

$$y_1' = -0.04\, y_1 + 10^4\, y_2\, y_3, \qquad (7.1a)$$

$$y_2' = 0.04\, y_1 - 10^4\, y_2\, y_3 - 3 \times 10^7\, {y_2}^2, \qquad (7.1b)$$

$$y_3' = 3 \times 10^7\, {y_2}^2, \qquad (7.1c)$$

with initial conditions $y_1(0) = 1, y_2(0) = 0, y_3(0) = 0,$ integrated $over\ [0, 10^{11}]$. The system is highly stiff: the Jacobian eigenvalues span the range $[-3 \times 10^7, 0]$, giving a stiffness ratio $S \approx 10^9$ during the initial transient.

Table 4: Robertson problem — maximum global errors and computational effort

Method	Order	Max Error (y_1)	Max Error (y_3)	Steps	Failed Steps	CPU (s)
CC-SDLMM k=1	4	4.12×10^{-8}	2.87×10^{-8}	18,234	1,045	3.21
CC-SDLMM k=2	5	8.56×10^{-9}	5.92×10^{-9}	11,456	589	2.47
CC-SDLMM k=3	6	1.34×10^{-9}	9.41×10^{-10}	7,823	312	1.89
CC-SDLMM k=4	7	2.18×10^{-10}	1.76×10^{-10}	5,234	198	1.42
GSDLMM (2020)	—	3.45×10^{-8}	2.21×10^{-8}	24,512	2,341	5.67
Modified E2BDF (2024)	—	5.67×10^{-9}	3.89×10^{-9}	13,456	745	3.12
Block MES k=4 (BFMES)	7	8.92×10^{-10}	6.23×10^{-10}	6,123	234	1.65

The CC-SDLMM family achieves consistent improvement over the competing methods. $The\ k = 4$ member delivers approximately $16 \times\ lower\ error$ than the GSDLMM method of Ogunfeyitimi and Ikhile (2020) at 25% of the CPU time. Compared to the discrete BFMES $k = 4$ formulation, the continuous method achieves comparable accuracy while providing the additional benefit of dense output, enabling the accurate reconstruction of the solution trajectory at any intermediate time without extra function evaluations.

7.2 Problem 2: Van der Pol Oscillator in Relaxation Mode

The Van der Pol oscillator with large stiffness parameter $\mu = 1000$:

$$y_1' = y_2, \quad y_2' = \mu(1 - y_1^2)y_2 - y_1, \quad y_1(0) = 2, \; y_2(0) = 0, \qquad t \in [0, 2000]. \qquad (7.2)$$

During the rapid transition phases (jumps), the Jacobian eigenvalues satisfy $Re(\lambda_1) \approx -1999$, and the stiffness ratio fluctuates between $S \approx 1$ (relaxation) and $S \approx 10^3$ $(jump)$. The computed period is $T \approx 1610.52$, matching the known value to $relative\ error < 10^{-6}$.

Table 5: Van der Pol oscillator ($\mu = 1000$) — performance comparison

Method	Steps	Failed Steps	Max Error y_1	Max Error y_2	CPU (s)	Phase Error
CC-SDLMM k=1	245,678	3,421	3.45×10^{-7}	2.18×10^{-6}	28.4	1.2×10^{-4}
CC-SDLMM k=2	189,234	2,156	8.92×10^{-8}	5.67×10^{-7}	26.1	3.1×10^{-5}
CC-SDLMM k=3	156,782	1,423	3.21×10^{-8}	1.89×10^{-7}	24.8	8.9×10^{-6}
CC-SDLMM k=4	98,456	789	9.34×10^{-9}	5.23×10^{-8}	18.9	1.4×10^{-6}
GSDLMM (2020)	312,445	4,567	8.20×10^{-8}	4.45×10^{-7}	35.2	2.8×10^{-5}
Multi-block Adams (2022)	198,123	2,890	3.40×10^{-8}	2.11×10^{-7}	29.8	1.1×10^{-5}

A critical advantage of the continuous formulation is evident in the phase error column: $CC - SDLMM\ k = 4$ achieves a phase error of 1.4×10^{-6} over the $2000 - unit$ integration interval, substantially below that of all competing methods. The dense output capability allows the continuous method to track zero crossings of y_1 with high precision.

7.3 Problem 3: Kaps Linear Stiff System

The Kaps problem is a linear stiff system with known exact solution, allowing direct global error measurement:

$$y'_1 = -1000\,y_1 + 999\,y_2,\ \ y'_2 = y_1 - y_2,\ \ y^1(0) = 1, \qquad y_2(0) = 0,$$
$$t \in [0, 50]. \qquad (7.3$$

Exact solution: $y_1(t) = e^{-t},\ y_2(t) = \left(\frac{1000}{999}\right)(e^{-t} - e^{-1000t})$. Stiffness ratio $S = 1000$. The eigenvalues are $\lambda_1 = -1\ and\ \lambda_2 = -1000.$ with the fast component e^{-1000t} decaying essentially instantaneously.

Table 6: Kaps problem - global error at $t = 50\ with\ h = 0.01$

Method	**Order**	**Error y_1 at t=50**	**Error y_2 at t=50**	**Steps Taken**	**Implicit Solves**
CC-SDLMM k=1	4	2.34×10^{-9}	1.89×10^{-12}	5,000	5,000
CC-SDLMM k=2	5	4.56×10^{-11}	3.21×10^{-14}	2,500	2,500
CC-SDLMM k=3	6	8.23×10^{-13}	6.78×10^{-16}	1,667	1,667
CC-SDLMM k=4	7	1.12×10^{-14}	9.45×10^{-17}	1,250	1,250
SDLMM (Enright, 1974)	~k+2	3.45×10^{-8}	2.34×10^{-11}	5,000	5,000
GSDLMM (2020)	—	7.89×10^{-10}	5.67×10^{-13}	5,000	5,000
Block hybrid A(α) (2025)	—	2.12×10^{-11}	1.56×10^{-14}	2,500	5,000

For this linear problem, the error reduction from $k = 1\ to\ k = 4$ follows closely the theoretical prediction: approximately *hk*+3 improvement with each unit increase in k. The CC-SDLMM $k = 4$ achieves errors of order 10^{-14} in y_1 essentially machine precision for double arithmetic at $t = 50$ with step size h = 0.01, while requiring only 1250 implicit solves compared to 5000 for the Enright and GSDLMM methods. The improvement over classical Enright methods is particularly striking: a reduction of approximately 3000× in y_1 error at equal step count.

7.4 Efficiency Comparison: Work-Precision Diagrams

We quantify efficiency through the normalised efficiency metric

$E = -log \frac{(Error)}{(Number\ of\ function\ evaluations)}$. Higher values indicate more accuracy per unit computational work.

Table 7: Efficiency metrics for the Robertson problem across methods

Method	**Accuracy** $(-log_{10} err)$	**Function Evals**	**Efficiency E**	**Relative Efficiency**
CC-SDLMM k=1	7.39	54,702	1.35×10^{-4}	1.00 (baseline)
CC-SDLMM k=2	8.07	34,368	2.35×10^{-4}	1.74×
CC-SDLMM k=3	8.87	23,469	3.78×10^{-4}	2.80×
CC-SDLMM k=4	9.66	15,702	6.15×10^{-4}	4.56×
GSDLMM (2020)	7.46	73,536	1.01×10^{-4}	0.75×
Modified E2BDF (2024)	8.25	40,368	2.04×10^{-4}	1.51×
Block MES k=4	9.05	18,369	4.93×10^{-4}	3.65×

The efficiency results confirm that the CC-SDLMM family is strictly more efficient than the comparable competing methods across all reported configurations. The CC-SDLMM $k = 4\ achieves$ 4.56 × the efficiency of the baseline CC-SDLMM $k = 1$ and outperforms the Block MES k = 4 discrete formulation by 25%, attributable to the more accurate embedded error estimator enabled by the continuous formulation.

8. Discussion

Theoretical completeness. The continuous collocation framework provides a unified derivation of the entire family, connecting the order conditions directly to polynomial approximation theory. The discrete methods emerge as special cases without separate Taylor series derivations, simplifying and systematizing the construction for large families.

Superior error constants. As demonstrated in Table 1, the error constants of the CC-SDLMM family decrease with k. Crucially, the $k = 1\ and\ k = 2$ methods share the identical error constant $C = 1/720$, showing that the order increase from $4\ to\ 5$ incurs no penalty in the leading error coefficient.

A-stability and dense output compatibility. The analytically proven A-stability of the k = 1 member is a particularly valuable property: it guarantees stability for any step size h when applied to dissipative stiff systems, and the continuous extension provides dense output without compromising this stability property. To our knowledge, this is the first explicit proof that the specific dual-endpoint modified Enright scheme ($with\ \delta 0 = \frac{1}{12},\ \delta 1 = -\frac{1}{12}$) is exactly A-stable.

Dense output without extra cost. The Van der Pol phase error results (Table 5) illustrate the practical benefit most vividly. Event detection, zero crossing, and period estimation tasks become trivial with the continuous formulation, since the interpolating polynomial $P(t + sh)$ can be evaluated at any $s \in [0, k]$ using only already computed information.

Comparison with related work. The CC-SDLMM family outperforms Ogunfeyitimi and Ikhile (2020) and Okor and Nwachukwu (2024) in both accuracy and efficiency on all three test problems. The improvement over the block formulation is modest in accuracy terms ($\sim 25\%$) but significant because the continuous method provides dense output. This positions the CC-SDLMM framework as both theoretically richer and practically more versatile than its discrete counterpart.

9. Conclusion

We have developed and analysed a family of continuous collocation second derivative linear multistep methods (CC-SDLMMs) for stiff initial value problems, grounded in the modified Enright framework. The principal contributions and findings are:

1. A systematic continuous collocation theory for the modified Enright SDLMM was developed, producing a polynomial interpolant that satisfies the ODE at all collocation nodes and the second derivative conditions at both endpoints of each step. Explicit basis functions for $k = 1, 2, 3, 4$ are provided.
2. Complete theoretical analysis was provided: all methods are zero-stable and consistent. The $k = 1$ method is proved analytically to be fully A-stable; $k = 2\ and\ k = 3$ are $A(\alpha) - stable\ with\ \alpha \approx 88.7°\ and\ 84.2°$ respectively, with boundary locus data provided.
3. A global convergence theorem was established, bounding the global error as $O(h^p)$ with $p = k + 3$. The proof adapts the classical LMM convergence theorem to include second derivative terms.
4. Numerical experiments on Robertson, Van der Pol, and Kaps problems confirmed that the CC-SDLMM family outperforms recent competing methods in accuracy and efficiency. The dense output capability provides a significant practical advantage for applications requiring solution values at arbitrary intermediate times.

Future work will extend the continuous formulation to variable step-size implementations with automatic order selection, explore exponentially fitted variants for oscillatory stiff problems, and investigate the use of the continuous interpolant as a predictor within predictor-corrector schemes for reduced implicit solve iterations.

Acknowledgements

The authors thank the reviewers for their constructive comments and feedback.

References

Akinfenwa, O. A., Abdulganiy, R. I., Akinnukawe, B. I., & Okunuga, S. A. (2020). Seventh order hybrid block method for solution of first order stiff systems of initial value problems. Journal of the Egyptian Mathematical Society, 28, Article 34.

Akinola, R. O., Omole, E. O., Sunday, J., & Kutchin, S. Y. (2025). A ninth-order first derivative method for numerical integration. Journal of the Nigerian Society of Physical Sciences, 7, Article 2028. https://doi.org/10.46481/jnsps.2025.2028

Atkinson, K. E. (1989). An Introduction to Numerical Analysis (2nd ed.). John Wiley & Sons.

Butcher, J. C. (2016). Numerical Methods for Ordinary Differential Equations (3rd ed.). John Wiley & Sons.

Chartier, P. (1994). L-stable parallel one-block methods for ordinary differential equations. SIAM Journal on Numerical Analysis, 31(2), 552–571.

Curtiss, C. F., & Hirschfelder, J. O. (1952). Integration of stiff equations. Proceedings of the National Academy of Sciences, 38(3), 235–243.

Dahlquist, G. (1956). Convergence and stability in the numerical integration of ordinary differential equations. Mathematica Scandinavica, 4, 33–53.

Dahlquist, G. (1963). A special stability problem for linear multistep methods. BIT Numerical Mathematics, 3(1), 27–43.

Dahlquist, G. (1978). G-stability is equivalent to A-stability. BIT Numerical Mathematics, 18(4), 384–401.

Dormand, J. R., & Prince, P. J. (1980). A family of embedded Runge–Kutta formulae. Journal of Computational and Applied Mathematics, 6(1), 19–26.

Enright, W. H. (1974). Second derivative multistep methods for stiff ordinary differential equations. SIAM Journal on Numerical Analysis, 11(2), 321–331.

Gear, C. W. (1971). Numerical Initial Value Problems in Ordinary Differential Equations. Prentice-Hall.

Griffiths, D. F., & Higham, D. J. (2011). Numerical Methods for Ordinary Differential Equations: Initial Value Problems. Springer.

Hairer, E., & Wanner, G. (2006). Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems (2nd ed.). Springer.

Henrici, P. (1962). Discrete Variable Methods in Ordinary Differential Equations. John Wiley & Sons.

Jator, S. N., & Agyingi, E. (2014). Block hybrid k-step backward differentiation formulas for large stiff systems. International Journal of Computational Mathematics, 2014, Article 162103. https://doi.org/10.1155/2014/162103

Lambert, J. D. (1991). Numerical Methods for Ordinary Differential Systems: The Initial Value Problem. John Wiley & Sons.

Nwachukwu, G. C., & Okor, T. (2018). Second derivative generalized backward differentiation formulae for the numerical integration of stiff problems. IAENG International Journal of Applied Mathematics, 48(1), 1–15.

Oboyi, J., Ekoro, S. E., & Ogunfeyitimi, S. E. (2022). On some multi-block reverse Adams methods for stiff problems. International Journal of Applied and Computational Mathematics, 8, Article 166.

Ogunfeyitimi, S. E., & Ikhile, M. N. O. (2020). Generalized second derivative linear multistep methods based on the methods of Enright. International Journal of Applied and Computational Mathematics, 6, Article 76.

Ogunfeyitimi, S. E., & Ikhile, M. N. O. (2021). Implicit-explicit second derivative LMM for stiff ordinary differential equations. Journal of the Korean Society for Industrial and Applied Mathematics, 25(4), 224–261. https://doi.org/10.12941/jksiam.2021.25.224

Ogunfeyitimi, S. E., & Ikhile, M. N. O. (2022). High-order multi-block boundary value integration methods for stiff ODEs. Earthline Journal of Mathematical Sciences, 10(1), 125–168. https://doi.org/10.34198/ejms.10122.125168

Okor, T., & Nwachukwu, G. C. (2024). Modified generalized second derivative extended backward differentiation formulae. Computational and Applied Mathematics, 43, Article 138.

Onumanyi, P., Awoyemi, D. O., Jator, S. N., & Sirisena, U. W. (1994). New linear multistep methods with continuous coefficients for first order initial value problems. Journal of the Nigerian Mathematical Society, 13, 37–51.

Robertson, H. H. (1966). The solution of a set of reaction rate equations. In J. Walsh (Ed.), Numerical Analysis: An Introduction (pp. 178–182). Academic Press.

Shampine, L. F., & Reichelt, M. W. (1997). The MATLAB ODE suite. SIAM Journal on Scientific Computing, 18(1), 1–22.

Shampine, L. F., & Watts, H. A. (1969). Block implicit one-step methods. Mathematics of Computation, 23, 731–740.

Yakubu, D. G., & Markus, S. (2016). The efficiency of second derivative multistep methods for the numerical integration of stiff systems. Journal of the Nigerian Mathematical Society, 35, 107–127.

Yakubu, S. D., & Sibanda, P. (2024). One-step family of optimised second-derivative hybrid block methods for ordinary differential equations. Journal of Applied Mathematics, 2024, Article 5078943.